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Transition rate for process involving particles with high momentum in a plasma and infrared physics for QED plasma

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Abstract

We derive a formula for computing the transition rate for a process involving particles with momentum much higher than the temperature and chemical potentials in a plasma by using an effective field theory approach. We apply it to collision of charged particles with hard momentum inside a QED plasma. The Debye screening effect and the damping of a charged particle moving in QED plasma are studied. Using the Bloch-Nordsieck resummation, the infrared divergences due to the absence of magnetic screening for QED plasma are shown not to appear in physically measurable rates. The soft plasmon absorption and emission for charged particles are discussed.

I. INTRODUCTION

Effective field theory has been proved to be powerful in dealing with different energy scales. Consider a process that contains particles with hard momentum (hard momentum means one of the component of the four momentum is larger than the temperature and the chemical potentials.) happening inside of a plasma. There are two relevant energy scales: the hard momentum scale and the soft momentum scale which is the thermal scale, *i.e.* order of the temperature T or the chemical potential. It is convenient to first "integrate out" the hard field degree of freedom with the momentum scale much higher than the temperature and the chemical potentials and end up with an effective action for the soft field degree of freedom. Taking the thermal average of the soft field degree of freedom yields the transition rate for the process. In this way, we can deal with the hard momentum scale and the thermal scale separately. Applying this formula to the hard process involving charged particle inside a QED plasma automatically incorporates the Debye screening effect and enables us to study the infrared physics for QED plasma due to the exchange of soft photon between the charged particles and the plasma, such as the damping of a charged particle moving in QED plasma as well as the emission and absorption of soft plasmon.

It is well known that the infrared divergences coming from the radiative correction due to the virtue soft photon are canceled by the real soft photon emission [1,2]. When the system is heated to a finite temperature, it is even more sensitive to the infrared region for the photons because of the bosonic distribution factor $n(\omega) \sim T/\omega$. It is interesting then to examine the cancellation of the infrared divergences for QED at finite temperature where a large population of thermal soft photons due to the Bose distribution enhances both the emission and the absorption of the soft photons for the charged particles. In the literature, there has been study of this problem for the free thermal photon background [3,4]. Our formula enables us to provide an analysis of these emission and absorption processes for the interacting photons inside QED plasma. The interaction between photons and the plasma brings the Debye screening mechanism to shield the electric force but not the magnetic force.

Therefore, potential infrared problem could occur due to the exchange of soft magnetic photons. In fact, there have been many discussions on the infrared divergence for the damping rate of a charged particle moving in QED plasma [5]. As discussed later in section III C, this arises because the unscreened magnetic force does not allow the charged particle moving with a definite momentum. For a physically measurable damping rate, this infrared divergence is cut off by the momentum resolution of our experiment. In general, we show that the infrared divergences due to the unscreened magnetic force cancels out by using the Bloch-Nordsieck resummation. We also discuss the correction to the transition rate due to the soft plasmon emission and absorption. At the massless limit of the plasmon, we reproduce the previous result [6].

The organization of this paper is as following. In next section, we develop first a general formula for calculating the transition rate for a hard process in a thermal background. In section III, our formula is applied to process inside QED plasma to study the Debye screening effect, the damping of a charged particle moving inside QED plasma, and infrared physics of QED plasma. Conclusions are drawn in section IV.

II. THERMAL AVERAGE OF TRANSITION RATE

In this section, we derive a general formula for computing the transition rate for a process involving hard momentum particles with the appearance of a relatively low (compared with the hard momentum scale) temperature T thermal background. The interaction between the thermal background and the hard particles will be considered.

Let us denote the fields with hard momentum by $\phi_{\rm h}$ and the fields with soft momentum by $\phi_{\rm s}$. Since, we consider the case where the temperature is much lower than the hard momentum, for the hard momentum fields, it is valid to treat them as at zero temperature. The thermal averaged transition rates reads

$$\langle R\rangle = \frac{1}{T_{\rm t}}\frac{1}{Z}\sum_{m,n}|\langle f,m|i,n\rangle|^2 e^{-\beta(E_n-\sum_i \mu_i Q_i(n))} \tag{2.1}$$

where T_t is the total time for the interaction to take place and i, f label the initial and final states of the hard momentum particles respectively. m, n represent the energy eigenstates generated by the soft momentum fields $\phi_{\rm s}$. A thermal distribution factor has been included with $\beta = 1/T$, μ_i are the chemical potentials corresponding to conserved charges Q_i. E_n and $Q_i(n)$ are the eigenvalues of the Hamiltonian and the charge Q_i for state $|n\rangle$. Z is the partition function for the soft momentum fields

$$Z = Tr\left(e^{-\beta(H-\sum_i \mu_i Q_i)}\right) = \sum_n e^{-\beta(E_n - \sum_i \mu_i Q_i(n))} \,. \tag{2.2}$$

To be more specific, we shall consider n_i initial and n_f final hard momentum particles are involved. Let us use p'_f with $f = 1, 2, \cdots, n_f$ and p_i with $i = 1, 2, \cdots, n_i$ denote the momentum of the final and initial particles. Therefore,

$$\langle R\rangle = \frac{1}{T_{\rm t}}\frac{1}{V^{n_i}}\frac{1}{Z}\prod_{f=1}^{n_f}\int_{\Omega_f}\frac{d^3p'_f}{(2\pi)^3}\sum_{n,m}|\langle m, p'_1, p'_2, \cdots, p'_{n_f}|p_1, p_2, \cdots, p_{n_i}, n\rangle|^2 e^{-\beta(E_n - \sum_i \mu_i Q_i(n))} \tag{2.3}$$

where Ω_f are the phase space volume within which the detectors are hunting for the final products. V is the total volume of the spatial box in which the wave function of the initial and final states are normalized. Expression (2.3) has following structure: matrix element $\langle m, p'_1, p'_2, \cdots, p'_{n_f}|p_1, p_2, \cdots, p_{n_i}, n\rangle$ involves time evolution from far past to far future while its complex conjugate evolves from far future to far past; the thermal distribution factor $\exp\{-\beta(E_n - \sum_i \mu_i Q_i(n))\}$ causes an evolution in imaginary time by amount $i\beta$. A real time formalism for the thermal field theory [7] can conveniently incorporate this structure so that we can express it in terms of functional integrals defined along paths in complex time plane. Slicing the time evolutions into pieces by inserting complete sets of states as we usually do to derive the path integral formalism and using the reduction formula to write the matrix element in terms of Green's function, Eq. (2.3) may be expressed as

$$\langle R\rangle = \frac{1}{T_{\rm t}}\frac{1}{V^{n_i}}\prod_{f=1}^{n_f}\int_{\Omega_f}\frac{d^3p_f}{(2\pi)^3}\int_P \mathcal{D}\phi_{\rm s}$$
$$\times \int_{P_1}\mathcal{D}\phi_{\rm h}\prod_{f=1}^{n_f}\int d^3\mathbf{x}_f g_f^*(p'_f, x_f)\frac{1}{i}\overleftrightarrow{\partial}_{x_f^0}\phi_{\rm h}(x_f)\prod_{i=1}^{n_i}\int d^3\mathbf{x}_i f_i(p_i, x_i)\frac{1}{i}\overleftrightarrow{\partial}_{x_i^0}\phi_{\rm h}(x_i)$$

$$\times \int_{P_2} \mathcal{D}\phi_{\rm h} \prod_{i=1}^{n_i} \int d^3\mathbf{x}'_i f_i^*(p_i, x'_i) \left(-\frac{1}{i}\overleftrightarrow{\partial}_{x'^0_i}\right) \phi^*_{\rm h}(x'_i) \ \prod_{f=1}^{n_f} \int d^3\mathbf{x}'_f g_f(p'_f, x'_f) \left(-\frac{1}{i}\overleftrightarrow{\partial}_{x'^0_f}\right) \phi^*_{\rm h}(x'_f)$$
$$\times \exp\left\{iS_P[\phi_{\rm s}] + iS_{P_1}[\phi_{\rm h}, \phi_{\rm s}] + iS_{P_2}[\phi_{\rm h}, \phi_{\rm s}]\right\} \ . \tag{2.4}$$

Here the notations need to be explained. $f_i(p_i, x_i)$ and $g_f(p_f, x_f)$ are the wave functions for the initial and final single particle states respectively. The path integrals are along contours in complex time denoted as τ plane: contour P_1 starts from $\tau{=}{-}\infty$ to $\tau{=}{+}\infty{-}i\epsilon$; contour P_2 starts from $\tau{=}{+}\infty{-}i\epsilon$ to $\tau{=}{-}\infty{-}2i\epsilon$; contour P includes P_1, P_2, and another piece from $\tau{=}{-}\infty{-}2i\epsilon$ to $\tau{=}{-}\infty{-}i\beta$. Boundary conditions are $\phi_{\rm s}(\tau{=}{-}\infty, \mathbf{x}){=} \pm\, \phi_{\rm s}(\tau{=}{-}\infty{-}i\beta, \mathbf{x})$ for the soft bosonic $(+)$ and fermionic $(-)$ momentum fields while $\phi_{\rm h}(\tau{=}\pm\infty, \mathbf{x}) \to 0$ for hard momentum fields. Actions $S_P[\phi_{\rm s}]$, $S_{P_1}[\phi_{\rm h}, \phi_{\rm s}]$, and $S_{P_2}[\phi_{\rm h}, \phi_{\rm s}]$ are defined with the time integral being the contour integral in the complex time τ plane along P, P_1, P_2 respectively. $S_P[\phi_{\rm s}]$ represents the part of the action contains only $\phi_{\rm s}$ while $S_{P_1}[\phi_{\rm h}, \phi_{\rm s}]$ and $S_{P_2}[\phi_{\rm h}, \phi_{\rm s}]$ contains both the part containing only $\phi_{\rm h}$ and the interaction terms between $\phi_{\rm h}$ and $\phi_{\rm s}$. The time component of vectors x_i, x_f belong to path P_1 while the time component of vectors x'_i, x'_f belong to path P_2. This type of path integral is used usually in the real time formalism for thermal field theory [7].

From an effective field theory point of view, we can first integrate out the hard momentum degree of freedom. The formula above may be rewritten as

$$\langle R\rangle = \frac{1}{T_{\rm t}}\frac{1}{V^{n_i}} \prod_{f=1}^{n_f} \int_{\Omega_f} \frac{d^3p_i}{(2\pi)^3} \int_P \mathcal{D}\phi_{\rm s} e^{iS_P[\phi_{\rm s}]} \langle p'_1, p'_2, \cdots, p'_{n_f} | p_1, p_2, \cdots, p_{n_i}\rangle_{\phi_{\rm s}, P_1}$$
$$\times \langle p_1, p_2, \cdots, p_{n_i} | p'_1, p'_2, \cdots, p'_{n_f}\rangle_{\phi_{\rm s}, P_2} \tag{2.5}$$

or more compactly

$$\langle R\rangle = \frac{1}{T_t} \int_P \mathcal{D}\phi_{\rm s} \exp\left\{iS_P[\phi_{\rm s}]\right\} \langle f|i\rangle_{\phi_{\rm s}, P_1} \langle i|f\rangle_{\phi_{\rm s}, P_2} \ , \tag{2.6}$$

where $\langle f|i\rangle_{\phi_{\rm s}, P_1}$ represents the matrix element between the hard momentum states $|i\rangle$ and $\langle f|$ with the background soft momentum field $\phi_{\rm s}$ appearing. Subscripts P_1, P_2 denote the time argument in field $\phi_{\rm s}$ belonging to contours P_1, P_2 respectively. We note that the thermal effect has been taken into account by the "strange" path integral that we use and the chemical

potential for the conserved charges are incorporated by including them in the action $S_P[\phi_{\rm s}]$ as the Lagrangian multipliers. It is convenient to introduce source terms for field $\phi_{\rm s}$ with time argument belonging to paths P_1 and P_2 respectively and define following generating functional

$$e^{iW[j_{\rm s},j'_{\rm s}]} \equiv \int_P \mathcal{D}\phi_{\rm s} \exp\left\{ iS_P[\phi_{\rm s}]+i\int_{P_1} d\tau \int d\mathbf{x} j_{\rm s}(\tau,\mathbf{x})\phi_{\rm s}(\tau,\mathbf{x})+i\int_{P_2} d\tau' \int d\mathbf{x}' j'_{\rm s}(\tau',\mathbf{x}')\phi^*_{\rm s}(\tau',\mathbf{x}') \right\} . \tag{2.7}$$

With this,

$$\langle R \rangle = \frac{1}{T_{\rm t}}\, \mathcal{A}_{fi} \left[\frac{1}{i}\frac{\delta}{\delta j_{\rm s}}\right] \mathcal{A}^*_{if} \left[\frac{1}{i}\frac{\delta}{\delta j'_{\rm s}}\right] e^{iW[j_{\rm s},j'_{\rm s}]}\Bigg|_{j_{\rm s}=0,j'_{\rm s}=0} , \tag{2.8}$$

where the functional $\mathcal{A}_{fi}[\phi_{\rm s}]$ is defined as

$$\mathcal{A}_{fi}[\phi_{\rm s}] \equiv \langle f|i\rangle_{\phi_{\rm s}} . \tag{2.9}$$

When the argument for $\mathcal{A}_{fi}$ is a functional derivative operator, the power series expansion of the functional is understood. We note that the path integral along contour P is simply another way to write the thermal average. Generating functional (2.7) may be written as a thermal average in an operator form:

$$e^{iW[j_{\rm s},j'_{\rm s}]}=\frac{1}{Z}Tr\left[e^{-\beta(H-\sum_i \mu_i Q_i)} T_+\left(e^{i\int_{-\infty}^{\infty}dt\int d\mathbf{x} j_{\rm s}(t,\mathbf{x})\phi_{\rm s}(t,\mathbf{x})}\right) T_-\left(e^{-i\int_{-\infty}^{\infty}dt'\int d\mathbf{x}' j_{\rm s}(t',\mathbf{x}')\phi^\dagger_{\rm s}(\mathbf{x}',t')}\right)\right] \tag{2.10}$$

where T_+ means time-ordered product according to t and T_- means anti-time-ordered product according to t'. Here the Hamiltonian H and the charges Q_i are operators for the degree of freedom corresponding to field operator $\phi_{\rm s}$.

The formula above enables us to deal with the two scales involved in the process separately. The characteristic energy scale for amplitude $\langle f|i\rangle_{\phi_{\rm s},P_1}$ is the hard momentum scale. We can then treat $\phi_{\rm s}$ as a slowly variant background field which enables us to use, for example, the semi-classical method. After we completed the hard momentum scale, we only need to deal with the soft momentum scale physics left.

We note that it does not require the momentum transfer to be big to get the formula above. All we need is that the particles involved has high momentum so that the thermal bath does not contain a noticeable amount of these hard momentum particles which we want to scatter and detect. This formula is also valid for the case where we just formed a plasma which is weakly coupled to another kind of particle, for example, the neutrinos. The plasma does not have enough time to produce the thermo-neutrinos, so that we can treat the neutrino fields the same way as $\phi_{\rm h}$ without requiring neutrinos to carry only hard momentum because the neutrino field is effectively at zero temperature. Practically, we can do a perturbation in the weak coupling constant to evaluate $\langle f|i\rangle_{\phi_{\rm s}}$ for using this formulation.

A concrete formulation for computing non-relativistic reaction rate has been well developed in reference [8].

III. PROCESS FOR HARD CHARGED PARTICLES IN QED PLASMA

We now apply the formulation in previous section to study explicitly a collision involving hard charged particles with the appearance of QED plasma. Let the process start with n_i initial particles and end up with n_f final particles plus soft photons. We assume that the initial particles have the four-momentum p_i, with $i = 1, 2, \cdots, n_f$ and we detect the final particles with momentum p'_f with $f = 1, 2, \cdots, n_f$ but left the soft photons undetected. In view of the formula in the previous section, $\phi_{\rm s}$ now represents the thermal photon and thermal electron fields while $\phi_{\rm h}$ represents the hard charged particles. Integrating out the degree of freedom corresponding to $\phi_{\rm h}$ to get an effective functional integrand for $\phi_{\rm s}$ means to get the zero temperature transition rate for hard charged particles with the appearance of the thermal QED background field. Expression (2.6) for transition rate reads

$$\begin{aligned}\langle R\rangle = &\frac{1}{T_{\rm t}}\frac{1}{V^{n_i}}\int_P \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{iS_P^{\rm eff}[A,\bar{\psi},\psi]\right\}\\ &\times\prod_{f=1}^{n_f}\int dy_f\phi_{\rm h}^{0\,*}(p'_f,y_f)(\partial_{y_f}^2+m^2)\prod_{i=1}^{n_i}\int dx_i\phi_{\rm h}^{0}(p_i,x_i)(\partial_{x_i}^2+m^2)\end{aligned}$$

$$
\times \prod_{i=1}^{n_i} \int dx_i' \phi_{\rm h}^{0*}(p_i, x_i')(\partial_{x_i'}^2 + m^2) \prod_{f=1}^{n_f} \int dy_f' \phi_{\rm h}^{0}(p_f', y_f')(\partial_{y_f'}^2 + m^2)
$$
$$
\times G[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A_{P_1}] G^*[x_1', x_2', \cdots, x_{n_i}', y_1', y_2', \cdots, y_{n_f}'; A_{P_2}] \,. \quad (3.1)
$$

Here $G[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A]$ is the Green's function for the hard particle fields with background photon field A. A_{P_1} and A_{P_2} denote the photon fields with the time argument being on the contours P_1 and P_2 respectively. $\phi_{\rm h}^0(p, x)$ is the wave function for the charged particle moving freely. The integration of the hard fields has two effects: 1) it gives the Green's function $G[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A]$; 2) it gives radiative corrections to the action for the photon and electron fields, i.e. we have an effective action. Except the effect of renormalization, these corrections are usually small because the high energy mode decouples [9].

Since $G[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A]$ contains only photon field, it is convenient to further integrate out the electron degree of freedom. Introducing source terms for the photon fields on paths P_1 and P_2, formally, we arrive at

$$
\begin{aligned}
\langle R \rangle = \frac{1}{T_{\rm t}} \frac{1}{V^{n_i}} & \prod_{f=1}^{n_f} \int dy_f \phi_{\rm h}^{0*}(p_f', y_f)(\partial_{y_f}^2 + m^2) \prod_{i=1}^{n_i} \int dx_i \phi_{\rm h}^{0}(p_i, x_i)(\partial_{x_i}^2 + m^2) \\
& \times \prod_{i=1}^{n_i} \int dx_i' \phi_{\rm h}^{0*}(p_i, x_i')(\partial_{x_i'}^2 + m^2) \prod_{f=1}^{n_f} \int dy_f' \phi_{\rm h}^{0}(p_f', y_f')(\partial_{y_f'}^2 + m^2) \\
& \times G\left[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; \frac{1}{i}\frac{\delta}{\delta J}\right] \\
& \times G^*\left[x_1', x_2', \cdots, x_{n_i}', y_1', y_2', \cdots, y_{n_f}'; \frac{1}{i}\frac{\delta}{\delta J'}\right] e^{iW[J,J']}\Big|_{J=0, J'=0} \,,
\end{aligned} \quad (3.2)
$$

where the generating functional $W[J, J']$ is defined by

$$
\begin{aligned}
\exp\{iW[J, J']\} &\equiv \int_P \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left\{ i S_P^{\rm eff}[A, \bar{\psi}, \psi] + i \int_{P_1} J \cdot A + i \int_{P_2} J' \cdot A \right\} \\
&= \left\langle T_+ \left(e^{i \int dt \int d\mathbf{x} J(t,\mathbf{x}) \cdot A(t,\mathbf{x})} \right) T_- \left(e^{-i \int dt' \int d\mathbf{x}' J'(t',\mathbf{x}') \cdot A(t',\mathbf{x}')} \right) \right\rangle \,,
\end{aligned} \quad (3.3)
$$

where we have used $\langle \cdots \rangle$ to denote the thermal average. Power series expansion of $W[J, J']$ can be written as a quadratic term plus higher power order terms in source fields J, J':

$$
W[J, J'] = -\frac{1}{2} \int d\mathbf{x}_1 \int d\mathbf{x}_2 \Big[\int dt_1 \int dt_2 J_\mu(t_1, \mathbf{x}_1) J_\nu(t_2, \mathbf{x}_2) G_{\mu\nu}^{(+)}(t_1, \mathbf{x}_1; t_2, \mathbf{x}_2)
$$

$$+\int dt_1'\int dt_2' J_\mu'(t_1',\mathbf{x}_1)J_\nu'(t_2',\mathbf{x}_2)G^{(-)}_{\mu\nu}(t_1',\mathbf{x}_1;t_2',\mathbf{x}_2)$$
$$-2\int dt_1\int dt_2' J_\mu(t_1,\mathbf{x}_1)J_\nu'(t_2',\mathbf{x}_2)\mathcal{G}_{\mu\nu}(t_1,\mathbf{x}_1;t_2',\mathbf{x}_2)\Big]+\cdots \tag{3.4}$$

where

$$G^{(+,-)}_{\mu\nu}(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2)\equiv\langle T_{(+,-)}\left(A_\mu(t_1,\mathbf{x}_1)A_\nu(t_2,\mathbf{x}_2)\right)\rangle\,,$$
$$\mathcal{G}_{\mu\nu}(t_1,\mathbf{x}_1;t_2',\mathbf{x}_2)\equiv\langle A_\mu(t_1,\mathbf{x}_1)A_\nu(t_2',\mathbf{x}_2)\rangle\,. \tag{3.5}$$

Again, we use $+,-$ to denote the time-ordered and anti-time-ordered product. We have following relations

$$G^{(+,-)}_{\mu\nu}(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2)=\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\mathcal{G}_{\mu\nu}(t_{(>,<)},t_{(<,>)};\mathbf{k}) \tag{3.6}$$

where $t_>=\max\{t_1,t_2\}$ and $t_<=\min\{t_1,t_2\}$ and $\mathcal{G}_{\mu\nu}(t,t';\mathbf{k})$ is defined by

$$\mathcal{G}_{\mu\nu}(t,\mathbf{x};t',\mathbf{x}')=\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}_{\mu\nu}(t,t';\mathbf{k})\,. \tag{3.7}$$

The Green's function $\mathcal{G}_{\mu\nu}$ may be expressed in terms of the spectral density function:

$$\mathcal{G}_{\mu\nu}(t,t',\mathbf{k})=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\int\frac{d^3k}{(2\pi)^3}n(\omega)\rho_{\mu\nu}(\omega,\mathbf{k})e^{-i\omega(t-t')}\,, \tag{3.8}$$

where $\rho_{\mu\nu}(\omega,\mathbf{k})$ is the usual spectral density function and $n(\omega)$ is the Bose distribution factor

$$n(\omega)=\frac{1}{e^{\beta\omega}-1}\,. \tag{3.9}$$

We shall focus on two aspects due to plasma. The first is for the momentum transfer is comparable with the Debye screening length, we show that the Debye screening effect can be properly taken into account in our formalism. The second case that we shall study is the modification to the transition rate due to the soft photon.

A. Debye screening

For the reason of simplicity, we consider the collision of two spin $\frac{1}{2}$ particles and ignore the soft photon emission and absorption which will be considered later. We have particle 1

with charge q_1 and 2 with charge q_2 moving with momentum p_1 and p_2 initially, colliding with each other via EM interaction, and separating with momentum p'_1 and p'_2 finally. When the momentum transfer is comparable to the Debye length for QED plasma, we shall be able to probe the Debye screening effect. We do a perturbation theory in QED coupling constant. At the leading order,

$$\langle p'_1, p'_2 | p_1, p_2 \rangle_{A,P_1} = -q_1 q_2 \int dx_1 e^{-i(p_1 - p'_1)\cdot x_1} A_\mu(x_1)\, \Phi_0^{1*}(p'_1)\gamma^\mu \Phi_0^1(p_1) \times \int dx_2 e^{-i(p_2-p'_2)\cdot x_2} A_\nu(x_2)\, \Phi_0^{2*}(p'_2)\gamma^\nu \Phi_0^2(p_2) \tag{3.10}$$

where $\Phi_0^{1,2}$ are the wave functions of the particles 1 and 2 moving freely with specific momentum respectively. Expression (2.8) for the rate reads

$$\begin{aligned}\langle R\rangle = & \frac{1}{T_{\rm t}}\frac{1}{V^2}\int_{\Omega_1}\frac{d^3p'_1}{(2\pi)^3}\int_{\Omega_2}\frac{d^3p'_2}{(2\pi)^3}\int dx_1\int dx_2\int dx'_1\int dx'_2\, e^{i(p_1-p'_1)(x_1-x'_1)+i(p_2-p'_2)(x'_2-x_2)} \\ & \times q_1^2\, q_2^2\, \bar\Phi_0^1(p'_1)\gamma^\mu\Phi_0^1(p_1)\, \bar\Phi_0^1(p_1)\gamma^\nu\Phi_0^1(p'_1)\, \bar\Phi_0^2(p'_2)\gamma^\lambda\Phi_0^2(p_2)\, \bar\Phi_0^2(p_2)\gamma^\sigma\Phi_0^2(p'_2) \\ & \times \frac{\delta}{\delta J_\mu(x_1)}\frac{\delta}{\delta J_\lambda(x_2)}\frac{\delta}{\delta J'_\nu(x'_1)}\frac{\delta}{\delta J'_\sigma(x'_2)} e^{iW[J,J']}\Big|_{J=0,J'=0}\,,\end{aligned} \tag{3.11}$$

where $\Omega_{1,2}$ are phase space volumes for particles 1 and 2 and the usual notation $\bar\Phi \equiv \Phi^\dagger\gamma^0$ has been used. Since the four-point connected Green's function for photon fields is of higher order, we shall neglect it. Upon using Eq. (3.4),

$$\begin{aligned}\langle R\rangle = & \frac{1}{T_{\rm t}}\frac{1}{V^2}\int_{\Omega_1}\frac{d^3p'_1}{(2\pi)^3}\int_{\Omega_2}\frac{d^3p'_2}{(2\pi)^3}\int dx_1\int dx_2\int dx'_1\int dx'_2\, e^{i(p_1-p'_1)(x_1-x'_1)+i(p_2-p'_2)(x'_2-x_2)} \\ & \times q_1^2\, q_2^2\, \bar\Phi_0^1(p'_1)\gamma^\mu\Phi_0^1(p_1)\, \bar\Phi_0^1(p_1)\gamma^\nu\Phi_0^1(p'_1)\, \bar\Phi_0^2(p'_2)\gamma^\lambda\Phi_0^2(p_2)\, \bar\Phi_0^2(p_2)\gamma^\sigma\Phi_0^2(p'_2) \\ & \times \left[G^{(+)}_{\mu\lambda}(x_1,x_2)G^{(-)}_{\nu\sigma}(x'_1,x'_2) + \mathcal{G}_{\mu\nu}(x_1,x'_1)\mathcal{G}_{\lambda\sigma}(x_2,x'_2) + \mathcal{G}_{\mu\sigma}(x_1,x'_2)\mathcal{G}_{\lambda\nu}(x_2,x'_1)\right]\,.\end{aligned} \tag{3.12}$$

The second term in the last square bracket corresponds to the process with particle 1 and 2 scattering with the plasma independently. The third term is an enhancing term for the special case with particle 1 and 2 scattering with the plasma independently but emitting the photon with the same four-momenta. We shall consider only the part in the transition rate which involves the interaction between particle 1 and 2. This corresponds to the first term. Therefore,

$$\langle R\rangle_{\rm int} = \frac{1}{T_{\rm t}}\frac{1}{V^2}\int_{\Omega_1}\frac{d^3p_1'}{(2\pi)^3}\int_{\Omega_2}\frac{d^3p_2'}{(2\pi)^3}\, q_1^2\, q_2^2\, \bar\Phi_0^1(p_1')\gamma^\mu\Phi_0^1(p_1)\,\bar\Phi_0^1(p_1)\gamma^\nu\Phi_0^1(p_1')$$
$$\times\bar\Phi_0^2(p_2')\gamma^\lambda\Phi_0^2(p_2)\,\bar\Phi_0^2(p_2)\gamma^\sigma\Phi_0^2(p_2')\,|(2\pi)^4\delta(p_1{+}p_2{-}p_1'{-}p_2')|^2\, G^{(+)}_{\mu\lambda}(k)\, G^{(+)*}_{\nu\sigma}(k)\,. \quad (3.13)$$

where $k = p_1' - p_1$ is the momentum transfer and we have used

$$G^{(-)}_{\mu\nu}(-k) = G^{(+)*}_{\mu\nu}(k)\,. \quad (3.14)$$

The Debye screening effect due to the plasma is taken into accounted by using the full two-point photon field Green's function $G^{(+)}(k)$. To illustrate this, we consider a simple case which is a collision of heavy particles, for example, protons, moving non-relativistically. Letting $v_{\rm rel}$ be the relative velocity for defining the cross section and using

$$\bar\Phi_0^1(p_1')\gamma^\mu\Phi_0^1(p_1) \approx \delta_{\mu 0}\,\Phi_0^{1\dagger}(p_1')\Phi_0^1(p_1)\,, \quad \bar\Phi_0^2(p_2')\gamma^\mu\Phi_0^2(p_2) \approx \delta_{\mu 0}\,\Phi_0^{2\dagger}(p_2')\Phi_0^2(p_2)\,, \quad (3.15)$$

the cross section may be expressed as

$$\begin{aligned}\langle d\sigma\rangle &\approx \frac{q_1^2 q_2^2}{v_{\rm rel}}\int_{\Omega_1}\frac{d^3p_1'}{(2\pi)^3}\int_{\Omega_2}\frac{d^3p_2'}{(2\pi)^3}(2\pi)^4\delta(p_1{+}p_2{-}p_1'{-}p_2')\\ &\quad\times|\Phi_0^{1\dagger}(p_1')\Phi_0^1(p_1)|^2|\Phi_0^{2\dagger}(p_2')\Phi_0^2(p_2)|^2|G^{(+)}_{00}(k)|^2\\ &\approx \frac{q_1^2 q_2^2}{v_{\rm rel}}\int_{\Omega_1}\frac{d^3p_1'}{(2\pi)^3}\int_{\Omega_2}\frac{d^3p_2'}{(2\pi)^3}(2\pi)^4\delta(p_1 + p_2 - p_1' - p_2')\\ &\quad\times|\Phi_0^{1\dagger}(p_1')\Phi_0^1(p_1)|^2|\Phi_0^{2\dagger}(p_2')\Phi_0^2(p_2)|^2\frac{1}{(\mathbf{k}^2{+}m_{\rm D}^2)^2}\end{aligned} \quad (3.16)$$

where we have used to the approximated result:

$$D_{00}(k_0 = 0, \mathbf{k}) \approx \frac{1}{\mathbf{k}^2 + m_{\rm D}^2} \quad (3.17)$$

for $\mathbf{k}$ being small. Here $m_{\rm D}$ is the Debye mass. When the momentum transfer is comparable with the inverse of the Debye screening length, it probes Debye screening length in the way above.

B. Interaction between charged particle and the plasma via soft photon

In this subsection, we shall study the effect due to the emission and absorption of the soft photons with energy and momentum much less than the typical momentum transfer.

"Soft" photons mean photons with energy and momentum much less that the momentum transfer. For the hard photons interacting with the charged particles, we can treat them perturbatively because of the smallness of the QED coupling. However, since the soft photon causes infrared divergences for individual diagram, we need to sum up the leading infrared behavior [1,2]. The process is always accompanied by emission and absorption of a lot of soft photons. Treating it perturbatively does not describe the process properly. For the soft photon fields, we shall sum up the leading infrared contribution to all order by using the Bloch-Nordsieck semi-classical method.

The LSZ reduction formula enables us to extract the scattering matrix element from Green's function. Equivalently, we consider the large time behavior of the Green's function and find the part having the correct asymptotic wave function for the incoming and outgoing particles. Mathematically,

$$\prod_{f=1}^{n_f}\int dy_f \phi_{\mathrm{h}}^{0*}(p'_f, y_f)(\partial_{y_f}^2+m^2)\prod_{i=1}^{n_i}\int dx_i \phi_{\mathrm{h}}^{0}(p_i, x_i)(\partial_{x_i}^2+m^2)G[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A]$$
$$= \prod_{f=1}^{n_f}\int dy_f \phi_{\mathrm{h}}^{*}[p_f, y_f; A]\prod_{i=1}^{n_i}\int dx_i \phi_{\mathrm{h}}[p_i, x_i; A]\Gamma[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A] \qquad (3.18)$$

where $\phi_{\mathrm{h}}[p_i, x_i, A]$ and $\phi_{\mathrm{h}}[p_f, y_f, A]$ are the wave functions of the charged particle moving in the classical background A field. $\Gamma[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A]$ is the amputated Green's function for the external charged particles or the one-hard-momentum-particle-irreducible Green's function. Since we focus on the contribution due to the soft photons and the hard photon may be treated perturbatively, to the leading order, we can neglect the hard photon contribution. Equivalently, A carries only very small energy and momentum which are much less than the momentum transfer. It is then valid to neglect soft photon field A in $\Gamma[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A]$ as the leading infrared behavior is concerned since Γ describes a hard process [10]. Our major task now is to calculate $\phi_{\mathrm{h}}[p, x; A]$ for soft background A field by using classical current approximation.

For the reason of simplicity, let us first consider $\phi_{\mathrm{h}}[p, x; A]$ for a scalar charged particle with charge q satisfying the Klein-Gordon equation:

$$(D_\mu D^\mu + m^2)\phi_{\rm h}[p,x;A] = 0\,, \tag{3.19}$$

where the covariant derivative $D_\mu = \partial_\mu - iqA_\mu$. It is not hard to verify that, at the leading order, the solution may be expressed as [10]

$$\phi_{\rm h}[p,x,A] = e^{ipx}\exp\left\{iq\int_C dz_\mu A^\mu(z)\right\} \tag{3.20}$$

where C is a straight line ending at x_μ with slope p_μ/m *i.e.* C is the classical trajectory of the charged particle moving with speed $v_\mu = p_\mu/m$. It is generally true that the solution of a charged particle with any spin moving in slowly varying background A field has the form:

$$\phi_{\rm h}[p,x;A] = \phi^0_{\rm h}(p,x)\exp\left\{iq\int_C dz_\mu A^\mu(z)\right\} \tag{3.21}$$

with $\phi^0_{\rm h}(p,x)$ being the free solution since the effect due to spin is proportional to the field strength $F_{\mu\nu}$ which is small [11]. It is convenient to define a classical current

$$J_\mu(z) = qv_\mu\delta^{(3)}(\mathbf{z}-\mathbf{x}-\mathbf{v}(z_0-x_0)) \tag{3.22}$$

so that we can write

$$\phi_{\rm h}[p,x;A] = \phi^0_{\rm h}(p,x)\exp\left\{i\int dz J_\mu(z)A^\mu(z)\right\}. \tag{3.23}$$

Now the soft photon problem can be simplified as the interaction between classical current and the plasma. The underlying reason for this is because the Compton wave length of the charged particle is much smaller than the EM field's characteristic wave length. Thus, at the scale of the wave length of the EM field, the charged particle behaves like a classical particle. This observation is originally due to Bloch and Nordsieck [1].

Since $\Gamma[x_1,x_2,\cdots,x_{n_i},y_1,y_2,\cdots,y_{n_f};A]$ is the correlation function obtained by integrating over the hard momentum, it certainly looks local to the soft momentum. This means that for the purpose of treating the soft photon, we can regard it as proportional to a delta function for the coordinates. Let us use $y\equiv y_1$ denote the space-time point where the hard collision happens. Each charged particle generates a current. The total current is the sum of them. Let us denote the total current as $J^{(y)}$.

$$J^{(y)}_\mu(z) = \theta(y_0 - z_0) \sum_i q_i v^i_\mu \delta(\mathbf{z} - \mathbf{y} - \mathbf{v}^i(z_0 - y_0)) + \theta(z_0 - y_0) \sum_f q_f v^f_\mu \delta(\mathbf{z} - \mathbf{y} - \mathbf{v}^f(z_0 - y_0)) \quad (3.24)$$

and the Fourier transform $J_\mu(\omega, \mathbf{k})$ is simply

$$J^{(y)}_\mu(\omega, \mathbf{k}) = ie^{iky} \left(\sum_i q_i \frac{p^i_\mu}{p^i \cdot k + i\epsilon} - \sum_f q_f \frac{p^f_\mu}{p^f \cdot k - i\epsilon} \right), \quad (3.25)$$

where $k = (\omega, \mathbf{k})$. The conservation of $J^{(y)}_\mu$ is guaranteed by $\sum_i q_i = \sum_f q_f$. We now have,

$$\begin{aligned} &\prod_{f=1}^{n_f} \int dy_f \phi^*_{\mathrm{h}}[p_f, y_f; A] \prod_{i=1}^{n_i} \int dx_i \phi_{\mathrm{h}}[p_i, x_i; A] \Gamma[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; A] \\ &\approx \prod_{f=1}^{n_f} \int dy_f \phi^{0*}_{\mathrm{h}}(p_f, y_f) \prod_{i=1}^{n_i} \int dx_i \phi^0_{\mathrm{h}}(p_i, x_i) \Gamma[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; 0] e^{i\int dz J^{(y)} \cdot A}. \quad (3.26) \end{aligned}$$

By translation invariance, we define $\tilde{\Gamma}$ as

$$\begin{aligned} &\phi^{0*}_{\mathrm{h}}(p_1, y_1) \prod_{f=2}^{n_f} \int dy_f \phi^{0*}_{\mathrm{h}}(p_f, y_f) \prod_{i=1}^{n_i} \int dx_i \phi^0_{\mathrm{h}}(p_i, x_i) \Gamma[x_1, x_2, \cdots, x_{n_i}, y_1, y_2, \cdots, y_{n_f}; 0] \\ &\qquad \equiv e^{iqy_1} \tilde{\Gamma}(p^i_1, p^i_2, \cdots, p^i_{n_i}; p^f_1, p^f_2, \cdots, p^f_{n_f}) \quad (3.27) \end{aligned}$$

where q is the total four-momentum lost for the hard particles due to the interaction with the plasma, i.e.

$$q = \sum_{i=1}^{n_i} p^i - \sum_{f=1}^{n_f} p^f. \quad (3.28)$$

The average rate including the correction due to the soft photon is expressed as

$$\langle R \rangle \approx \frac{1}{T_{\mathrm{t}}} \frac{1}{V^{n_i}} \prod_{f=1}^{n_f} \int_{\Omega_f} \frac{d^3 p_f}{(2\pi)^3} |\tilde{\Gamma}(p^i_1, p^i_2, \cdots, p^i_{n_i}; p^f_1, p^f_2, \cdots, p^f_{n_f})|^2 \int dy dy' e^{iq(y-y')} I(y, y') \quad (3.29)$$

where I is

$$I(y, y') = e^{J^{(y)} \frac{\delta}{\delta J'}} e^{J^{(y')} \frac{\delta}{\delta J'}} \left. e^{iW[J, J']} \right|_{J=0, J'=0} = e^{iW[J^{(y)}, J^{(y')}]}. \quad (3.30)$$

with $J^{(y)}$ given by Eq. (3.24) and $J^{(y')}$ given by

$$J^{(y')}_\mu(z) = J^{(y)}_\mu(z - y' + y). \quad (3.31)$$

To the leading order, we only need the quadratic term in the expansion (3.4) for $W[J, J']$. $I(y, y')$ may be written as

$$I(y,y') = \exp\biggl\{ -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\int dt \int dt' \Bigl[J_\mu(t,\mathbf{k})J_\nu(t',-\mathbf{k})\mathcal{G}_{\mu\nu}(t_>,t_<,\mathbf{k}) + J'_\mu(t,\mathbf{k})J'_\nu(t',-\mathbf{k})\mathcal{G}_{\mu\nu}(t_<,t_>,\mathbf{k}) - 2J'_\mu(t,\mathbf{k})J_\nu(t',-\mathbf{k})\mathcal{G}_{\mu\nu}(t',t,\mathbf{k})\Bigr]\biggr\}. \tag{3.32}$$

Upon using the fact

$$J'_\mu(t,\mathbf{k}) = J_\mu(t-(y_0-y'_0),\mathbf{k})\, e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{y}')} \tag{3.33}$$

and

$$J'_\mu(\omega,\mathbf{k}) = J_\mu(\omega,\mathbf{k})\, e^{ik(y'-y)}\,, \tag{3.34}$$

we have

$$\begin{aligned} I(y,y') &= \exp\biggl\{ -i\int \frac{d^3k}{(2\pi)^3}\int dt\int dt' \Bigl[J_\mu(t,\mathbf{k}) - J'_\mu(t,\mathbf{k})\Bigr] J_\nu(t',-\mathbf{k})\mathcal{G}_{\mu\nu}(t',t,\mathbf{k})\biggr\} \\ &= \exp\biggl\{ -\int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3} J^*_\mu(\omega,\mathbf{k})J_\nu(\omega,\mathbf{k})n(\omega)\rho_{\mu\nu}(\omega,\mathbf{k})\left(1-e^{ik\cdot(y'-y)}\right)\biggr\}. \end{aligned} \tag{3.35}$$

Since current J is conserved, we can choose any gauge to work on. In Landau gauge, We shall use following decomposition to separate the longitudinal part and the transverse part of the spectral density function:

$$\rho_{\mu\nu}(\omega,\mathbf{k}) = \rho_{\rm t}(\omega,\mathbf{k})P^{\rm t}_{\mu\nu} + \rho_{\rm l}(\omega,\mathbf{k})P^{\rm l}_{\mu\nu} \tag{3.36}$$

where

$$\begin{aligned} P^{\rm t}_{\mu\nu} &= \delta_{\mu i}\left(\delta_{ij} - \frac{\mathbf{k}_i\mathbf{k}_j}{\mathbf{k}^2}\right)\delta_{j\nu} \\ P^{\rm l}_{\mu\nu} &= \left(\delta_{\mu 0} - \frac{\omega\mathbf{k}_\mu}{k^2}\right)\left(\delta_{\nu 0} - \frac{\omega\mathbf{k}_\nu}{k^2}\right)\frac{k^2}{\mathbf{k}^2}\,. \end{aligned} \tag{3.37}$$

Defining $\mathcal{J}_{\rm t,l}$

$$\mathcal{J}_{\rm t,l} \equiv J^*_\mu(\omega,\mathbf{k})\, P^{\rm t,l}_{\mu\nu}\, J_\nu(\omega,\mathbf{k}) \tag{3.38}$$

enables to express

$$I(y,y') = \exp\biggl\{ -\int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3} n(\omega)\left(\mathcal{J}_{\rm t}\,\rho_{\rm t} + \mathcal{J}_{\rm l}\,\rho_{\rm l}\right)\Bigl[1-e^{ik\cdot(y'-y)}\Bigr]\biggr\}, \tag{3.39}$$

Observing that $I(y, y')$ depends only on $y - y'$ in Eq. (3.29), the space-time integrals over y and y' gives the total volume of the space-time. Therefore,

$$\langle R\rangle = \frac{1}{V^{n_i-1}}\prod_{i=1}^{n_f}\int_{\Omega_i}\frac{d^3p_f}{(2\pi)^3}|\tilde{\Gamma}(p_1^i, p_2^i, \cdots, p_{n_i}^i; p_1^f, p_2^f, \cdots, p_{n_f}^f)|^2\int dy\, e^{-iqy}\, e^{-N(y)}\,. \tag{3.40}$$

where we have defined $N(y)$

$$N(y) \equiv \int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}n(\omega)\left(\mathcal{J}_{\rm t}\,\rho_{\rm t} + \mathcal{J}_{\rm l}\,\rho_{\rm l}\right)\left(1 - e^{ik\cdot y}\right)\,. \tag{3.41}$$

In principle, one should be able to use the expression (3.40) to compute the effect on the hard process due to its interaction with the plasma via soft photons. However, we shall not go to the detail of the computation but only focus on two aspects, namely, the infrared behavior and the absorption and emission of soft plasmon. Before generally discussing the infrared physics of formula (3.40), we like to study first the infrared behavior of the damping of a charged particle moving in QED plasma with hard momentum. The charged particle can either move very fast or be very heavy so that its momentum is hard.

C. Damping of a charged particle

The question we ask is what is the probability of detecting the charged particle moving with the initial momentum at time $T_{\rm t}$, *i.e.* the probability for finding the charged particle in the initial state. We expect the answer may be expressed as an exponentially decaying factor with the exponent proportional to the time $T_{\rm t}$. Thus the half of the proportional factor in front of the time $T_{\rm t}$ in the exponent is the damping rate. We also expect that the answer may depend on the momentum resolution by which we judge whether the particle's momentum is the same as the initial momentum since the soft photon emission and absorption processes always appear. We shall study the leading infrared contribution to the damping of the charged particle due to its interaction with the plasma via the exchange of soft photons. Let Ω, a small volume in momentum space, describe the resolution. By using the standard reduction technique, we can construct the probability for finding the charged particle to

have momentum $\mathbf{p}'$ lying in the phase space volume Ω centered at $\mathbf{p}$. Keeping the leading infrared terms, it may be expressed as

$$P = \int_\Omega \frac{d^3\mathbf{p}'}{(2\pi)^3} \int d\mathbf{y}\, e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{y}}\, e^{-N(0,\mathbf{y})} \tag{3.42}$$

where

$$N(0,\mathbf{y}) = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} n(\omega) J^*_\mu(\omega,\mathbf{k}) J_\nu(\omega,\mathbf{k}) \rho_{\mu\nu}(\omega,\mathbf{k}) \left(1 - e^{i\mathbf{k}\cdot\mathbf{y}}\right) \tag{3.43}$$

with the current

$$J_\mu(\omega,\mathbf{k}) = \int_0^{T_\mathrm{t}} dt \int d\mathbf{x}\, e^{i\omega t - i\mathbf{k}\mathbf{x}}\, q\, v_\mu \delta(\mathbf{x}-\mathbf{v}t) = q\, v_\mu \frac{e^{i(\omega-\mathbf{v}\cdot\mathbf{k})T_\mathrm{t}} - 1}{i(\omega - \mathbf{v}\cdot\mathbf{k})} \tag{3.44}$$

Noting that

$$J^*_\mu(\omega,\mathbf{k}) J_\nu(\omega,\mathbf{k}) \to q^2 v_\mu v_\nu T_\mathrm{t}\, (2\pi)\, \delta(\omega - \mathbf{v}\cdot\mathbf{k}) \quad \text{as} \;\; T_\mathrm{t} \to \infty \tag{3.45}$$

we can perform the integral over ω and get

$$N(0,\mathbf{y}) = q^2 T_\mathrm{t} \int \frac{d^3k}{(2\pi)^3} n(\mathbf{k}\cdot\mathbf{v}) v_\mu v_\nu \rho_{\mu\nu}(\mathbf{k}\cdot\mathbf{v},\mathbf{k}) \left(1 - e^{i\mathbf{k}\cdot\mathbf{y}}\right) . \tag{3.46}$$

We shall look at the leading infrared behavior. The spectral density $\rho_{\mu\nu}$ is dominated by its transverse part ρ_t at the infrared region. As $|\mathbf{k}|$ becomes small, the leading infrared behavior for ρ_t is [12]

$$\frac{\rho_t(\omega,\mathbf{k})}{\omega} \to \frac{1}{\mathbf{k}^2}(2\pi)\delta(\omega) . \tag{3.47}$$

Using this, we can complete the integral over the angular part for $\mathbf{k}$ integral as

$$\begin{aligned} N(0,\mathbf{y}) &= q^2 T\, T_\mathrm{t} \int_0^\infty e^{-k/\Lambda} \frac{dk}{4\pi^2 k} |\mathbf{v}| \int_0^{2\pi} d\phi \left(1 - e^{ik|\mathbf{y}|\sin\theta\cos\phi}\right) \\ &= q^2\, T\, T_\mathrm{t} \int_0^{2\pi} d\phi \frac{1}{4\pi^2} |\mathbf{v}| \ln(1 - i\Lambda|\mathbf{y}|\sin\theta\cos\phi) \\ &\approx \frac{q^2}{2\pi} T\, T_\mathrm{t}\, |\mathbf{v}| \ln(\Lambda|\mathbf{y}|\sin\theta) = \frac{q^2}{2\pi} T\, T_\mathrm{t}\, |\mathbf{v}| \ln(\Lambda|\mathbf{y}_\mathrm{t}|) , \end{aligned} \tag{3.48}$$

where an ultraviolet cutoff Λ for the k integral has been introduced, θ is the angle between $\mathbf{y}$ and the velocity of the charged particle $\mathbf{v}$, and $\mathbf{y}_\mathrm{t} \equiv \mathbf{y} - (\mathbf{y}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$. Therefore,

$$P \sim \left(\frac{\Delta p_{\rm t}}{\Lambda}\right)^{\frac{q^2}{2\pi}T|\mathbf{v}|T_{\rm t}} = \exp\left\{-\frac{q^2}{2\pi}T|\mathbf{v}|\ln\left(\frac{\Lambda}{\Delta p_{\rm t}}\right)T_{\rm t}\right\}\,, \tag{3.49}$$

where $\Delta p_{\rm t}$ is the transverse size of Ω with respect to the velocity $\mathbf{v}$ of the charged particle. At the leading order, the damping rate is

$$\Gamma_{\rm damping} = \frac{q^2}{4\pi}T|\mathbf{v}|\ln\left(\frac{\Lambda}{\Delta p_{\rm t}}\right)\,. \tag{3.50}$$

The damping rate depends on the momentum resolution $\Delta p_{\rm t}$ logarithmically due to the exchange of soft magnetic photon since the magnetic force is not screened in QED plasma and is transverse to $\mathbf{v}$. This leading piece of the damping rate vanishes if $\mathbf{v} = \mathbf{0}$ which is the case where the charged particle does not feel the magnetic force. If we take the limit $\Delta p_{\rm t} \to 0$, the damping rate becomes infinite. This means that the particle experiences the long range magnetic force and can not move with a definite momentum. There has been discussions about the infrared problem for the damping rate of charged particles moving in QED plasma in the literature [5]. This infrared problem can be resolved by considering only a physically measurable rate which introduces the momentum resolution $\Delta p_{\rm t}$ to cut off the infrared divergence.

To avoid confusion, we note that only the infrared behavior of the damping rate has been examined. Equation (3.50) is valid only when the cutoff Λ being larger than $\Delta p_{\rm t}$. Cutoff Λ is the scale below which the asymptotic behavior 3.47 can be used. At high temperature, $\Lambda \sim eT$ while at low temperature Λ is suppressed by factor $e^{-\beta m_{\rm e}}$ with $m_{\rm e}$ being the electron mass. Therefore, at temperature $T \ll m_{\rm e}$, this infrared cutoff becomes almost zero. Our resolution $\Delta p_{\rm t}$ will not be able to see this scale. Thus, the "leading infrared behavior" is essentially negligible. In view of Eq. (3.48), this corresponds to the case $\Lambda|\mathbf{y}| \ll 1$ so that $N(0,\mathbf{y})$ is small. These comments apply to discussions in next subsection as well.

D. General discussion of the infrared behavior

The cancellation of infrared divergences have been studied for the case of ideal thermal photon gas in reference [3]. This is the case where the temperature is low and the chemical

potential for the electron is small so that the photon gas is almost ideal. Here we shall provide a general discussion of the infrared physics. Using the result (3.25),

$$\mathcal{J}^{\mathrm{t,l}} = \left(\sum_i \frac{q_i\, p_{i\mu}}{p_i{\cdot}k{+}i\epsilon} - \sum_f \frac{q_f\, p_{f\mu}}{p_f{\cdot}k{-}i\epsilon} \right) \left(\sum_i \frac{q_i\, p_{i\nu}}{p_i{\cdot}k{-}i\epsilon} - \sum_f \frac{q_f\, p_{f\nu}}{p_f{\cdot}k{+}i\epsilon} \right) P^{\mathrm{t,l}}_{\mu\nu}\,. \tag{3.51}$$

Putting this into the definition (3.41) for integral $N(y)$, it is not hard to find that there are singularities in the integral over ω or $\mathbf{k}$ corresponding to pinching two poles together as $\epsilon \to 0$. The physical meaning of these singularities which come from putting the charged particle's propagator on-shell is that the charged particle suffers the damping as discussed in the previous subsection. Therefore, their contributions to $N(y)$ can be expressed similarly as the result shown in Eq. (3.48). We make following replacement to subtract out the damping effect of the single particle:

$$J^*_\mu(k) J_\nu(k) \to J^*_\mu(k) J_\nu(k) - \sum_i \frac{q_i^2\; p_{i\,\mu} p_{i\,\nu}}{(p_i \cdot k + i\epsilon)(p_i \cdot k - i\epsilon)} - \sum_f \frac{q_f^2\; p_{f\,\mu} p_{f\,\nu}}{(p_f \cdot k - i\epsilon)(p_f \cdot k + i\epsilon)}\,. \tag{3.52}$$

Because of Debye screening, the exchange of the magnetic photons dominate at the infrared region for the $\omega, \mathbf{k}$ integrals.. By the sum rule [12]

$$\int \frac{d\omega}{2\pi} \frac{\rho_{\mathrm{t}}(\omega, \mathbf{k})}{\omega} \sim \frac{1}{\mathbf{k}^2} \tag{3.53}$$

and the behavior

$$\frac{\rho_{\mathrm{t}}(\omega, \mathbf{k})}{\omega} \to \frac{1}{\mathbf{k}^2} (2\pi) \delta(\omega) \tag{3.54}$$

as $\mathbf{k} \to 0$, we have the leading infrared contribution to $N(y)$

$$N(y)_{\mathrm{infra}} \approx T \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2} \left(\mathbf{J}^*(0,\mathbf{k}){\cdot}\mathbf{J}(0,\mathbf{k}) - \sum_i \left| \frac{q_i}{\hat{\mathbf{v}}_i \cdot \mathbf{k}{+}i\epsilon} \right|^2 - \sum_f \left| \frac{q_f}{\hat{\mathbf{v}}_f \cdot \mathbf{k}{-}i\epsilon} \right|^2 \right) (1{-}e^{i\mathbf{k}\cdot\mathbf{y}}) \tag{3.55}$$

besides the contribution from the damping of the individual particles. The $\mathbf{k}$ integral in expression above may be written as a one-parameter integral as shown in the appendix.

Using the conservation of the current so as to replace $P^{\rm t}_{\mu\nu}$ by δ_{ij}, apart from the contribution due to the damping effect,

$$N(y)_{\rm infra} \approx T|\mathbf{y}| \left[\sum_{i=1}^{n_i} \sum_{i'=i+1}^{n_i} q_i q_{i'} I(\hat{\mathbf{y}}, \hat{\mathbf{v}}_i, -\hat{\mathbf{v}}_{i'}) + \sum_{f=1}^{n_f} \sum_{f'=f+1}^{n_f} q_f q_{f'} I(\hat{\mathbf{y}}, -\hat{\mathbf{v}}_f, \hat{\mathbf{v}}_{f'}) \right.$$
$$\left. + \sum_{i=1}^{n_i} \sum_{f=1}^{n_f} q_i q_f I(\hat{\mathbf{y}}, \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_f) \right] , \tag{3.56}$$

where $I(\hat{\mathbf{y}}, \mathbf{v}, \mathbf{v}')$ is defined and simplified in the appendix. Since $I(\hat{\mathbf{y}}, \mathbf{v}, \mathbf{v}')$ contains no infrared divergence, $N(y)_{\rm infra}$ is finite. The unscreened static magnetic force in QED plasma causes no infrared divergences.

E. long wave plasma oscillation

Besides to the leading infrared effect due to the unscreened static magnetic force, there are also effects due to the excitations of the long wave plasma oscillation. To simplify the discussion, let us isolate the contribution of the plasmon by considering

$$\rho_{\rm t,l} = (2\pi)\delta(\omega^2 - \omega_{\rm t,l}(\mathbf{k})^2) \tag{3.57}$$

with $\omega = \omega_{\rm t,l}(\mathbf{k})$ being the dispersion relation for the transverse and longitudinal plamons respectively. Putting this into the expression (3.41) gives

$$N(y)_{\rm pl} = \int \frac{d^3k}{(2\pi)^3} \left[\frac{\mathcal{J}_{\rm t}}{2\omega_{\rm t}(\mathbf{k})} \Big[(1 - e^{ik_{\rm t} y}) + 2n(\omega_{\rm t}(\mathbf{k}))(1 - \cos k_{\rm t} y) \Big] \right.$$
$$\left. + \frac{\mathcal{J}_{\rm l}}{2\omega_{\rm l}(\mathbf{k})} \Big[(1 - e^{ik_{\rm l} y}) + 2n(\omega_{\rm l}(\mathbf{k}))(1 - \cos k_{\rm l} y) \Big] \right] , \tag{3.58}$$

where $k_{\rm t,l} = (\omega_{\rm t,l}(\mathbf{k}), \mathbf{k})$. For the integrals in Eq. (3.58), the plasma mass $m_{\rm pl} \equiv \omega_{\rm t,l}(\mathbf{k} = \mathbf{0})$ cuts off the integral at the infrared region. At very high temperature $m_{\rm pl} \sim eT$ or at low temperature with sufficiently large chemical potential, $m_{\rm pl} \gg e^2 T$, $N(y)$ is of order $e^2 T / m_{\rm pl} \ll 1$, thus perturbation theory works well since we do not need to sum over the series to form the small exponential factor.

At low temperature and small chemical potential, $m_{\rm pl}$ may be much smaller than $e^2 T$. There are two cases that we would like to discuss.

1) For the case where the exchange of energy between the charged particles and the plasma is much smaller or the same order of magnitude of $m_{\rm pl}$, we then need to compute integral (3.58) by using, for example, the one-loop spectral density function. The calculation is rather involved. We leave it to another day.

2) For the case where the exchange of energy between the charged particles and the plasma is much larger than $m_{\rm pl}$, we can set $m_{\rm pl}$ to be zero and simply use the dispersion relation $\omega_{\rm t,l}(\mathbf{k}) = |\mathbf{k}|$. For this case, the contribution to $N(y)$ from the plasmon dominates since the photons are nearly ideal. Therefore,

$$N(y) \approx N(y)_{\rm pl} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}\,|J_\mu(|\mathbf{k}|,\mathbf{k})|^2\left[(1-e^{iky}) + 2n(|\mathbf{k}|)(1-\cos ky)\right]. \tag{3.59}$$

Writing

$$|J_\mu(|\mathbf{k}|,\mathbf{k})|^2 = \frac{1}{\mathbf{k}^2}\left|\sum_{i=1}^{n_i}\frac{q_i\,p_{i\mu}}{p_i\cdot u} - \sum_{f=1}^{n_f}\frac{q_f\,p_{f\mu}}{p_f\cdot u}\right|^2 \tag{3.60}$$

with four-vector $u \equiv (1,\hat{\mathbf{k}})$, we have

$$N(y) \approx \frac{1}{8\pi^3}\int d\hat{\mathbf{k}}\left|\sum_{i=1}^{n_i}\frac{q_i\,p_{i\mu}}{p_i\cdot u} - \sum_{f=1}^{n_f}\frac{q_f\,p_{f\mu}}{p_f\cdot u}\right|^2\int_0^\infty\frac{d\omega}{2\omega}\left[(1-e^{i\omega u\cdot y}) + \frac{2}{e^{\beta\omega}-1}(1-\cos\omega u\cdot y)\right]. \tag{3.61}$$

Expression (3.61) above contains an ultraviolet divergence which should be cut off at the hard momentum scale Λ. Instead of using a sharp cut off, we use an exponential factor $e^{-\omega/\Lambda}$ to suppress the contribution from the region with ω bigger than Λ. Using the result in the appendix for the ω integral,

$$N(y) \approx \frac{1}{16\pi^3}\int d\hat{\mathbf{k}}\left|\sum_{i=1}^{n_i}\frac{q_i\,p_{i\mu}}{p_i\cdot u} - \sum_{f=1}^{n_f}\frac{q_f\,p_{f\mu}}{p_f\cdot u}\right|^2\left[\ln(1+i\Lambda u\cdot y) + \ln\frac{\sinh \pi T u\cdot y}{\pi T u\cdot y}\right]. \tag{3.62}$$

A full analysis of expression (3.62) is still formidable. We shall consider a special case where the process involves at least one heavy particle moving non-relativistically so that it can absorb the extra momentum without affect much the conservation of energy. Thus, we effectively has only the energy conservation without worrying about where to dump the

extra momentum. For this case, we can effectively replace $\mathbf{y}$ by $\mathbf{0}$ and only left with the time component. It is convenient to consider the differential rate with the energy exchange with the plasmons in plasma to be in the interval $(E, E+dE)$:

$$
\begin{aligned}
d\langle R\rangle &\propto \int dt e^{-iEt} \exp\left\{ -A\left[\ln(1+i\Lambda t) + \ln\frac{\sinh \pi T t}{\pi T t}\right]\right\}\frac{dE}{2\pi} \\
&= \int dt e^{-iEt}(1+i\Lambda t)^{-A}\left(\frac{\sinh \pi T t}{\pi T t}\right)^{-A}\frac{dE}{2\pi}
\end{aligned} \tag{3.63}
$$

where A is defined by

$$
A \equiv \frac{1}{16\pi^3}\int d\hat{\mathbf{k}}\left(\sum_{i=1}^{n_i} q_i \frac{p_{i\mu}}{p_i\cdot u} - \sum_{f=1}^{n_f} q_f \frac{p_{f\mu}}{p_f\cdot u}\right)^2 . \tag{3.64}
$$

We can then reproduce two previous results [2,6]. For the case $T \ll E$, it is expected that the result is approximately the same as the zero temperature case. For the t integral in expression (3.63), typical t is of scale $1/E$ and thus $Tt \ll 1$. Noting $\sinh x \approx x$ for small x, we have

$$
d\langle R\rangle \propto \int dt e^{-iEt}(1+i\Lambda t)^{-A}\frac{dE}{2\pi} \sim dE\,\theta(E)\,\frac{A}{E}\left(\frac{E}{\Lambda}\right)^A . \tag{3.65}
$$

Therefore, the transition rate with soft photon carrying away energy no more than ΔE with $\Delta E \gg T$ is

$$
\langle R\rangle_{\Delta E} \propto \int_0^{\Delta E} dE\,\frac{A}{E}\left(\frac{E}{\Lambda}\right)^A = \left(\frac{\Delta E}{\Lambda}\right)^A . \tag{3.66}
$$

This is the zero temperature result [2].

For the case where the energy exchange between plasma and charged particles are much smaller than the temperature, $T \gg E$. Now for typical t in the integral in Eq. (3.63), $Tt \gg 1$. Using $\sinh x \sim e^x$ for large x,

$$
\begin{aligned}
d\langle R\rangle &\propto \frac{dE}{2\pi}\int dt e^{-iEt}e^{-\pi A T|t|}\left(\frac{T}{\Lambda}\right)^A \\
&= \frac{dE}{2\pi}\left(\frac{T}{\Lambda}\right)^A\left[\frac{1}{\pi AT+iE} + \frac{1}{\pi AT-iE}\right] \\
&\approx \frac{dE}{2\pi}\frac{2\pi AT}{E^2+(\pi AT)^2} .
\end{aligned} \tag{3.67}
$$

The agrees with previous result [6].

IV. CONCLUSIONS

A general formula for computing the transition rate for hard process happening in a soft thermal background is derived. In particular, we use it to study hard process involving charged particle inside QED plasma. Both the Debye screening and the damping of the hard charged particles are incorporated in our formula. We show that the transition rate is free of infrared divergence even though the magnetic force is not screened for QED plasma. Soft plasmon emission and absorption are analyzed. Our formalism can also be applied to the case where some or all of the hard momentum particles are replaced by particles which interact weakly with the plasma so that after the formation of the plasma, it does not have enough time to have thermal contact with the degree of freedom corresponding to these particles.

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APPENDIX A: COMPUTATION OF INTEGRALS

We first evaluate following integral:

$$I(\hat{\mathbf{y}},\mathbf{v},\mathbf{v}') \equiv \frac{1}{|\mathbf{y}|}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{v}\cdot\mathbf{k}+i\epsilon}\frac{1}{\mathbf{v}'\cdot\mathbf{k}+i\epsilon}\frac{1}{\mathbf{k}^2}\left[2-e^{i\mathbf{k}\cdot\mathbf{y}}-e^{-i\mathbf{k}\cdot\mathbf{y}}\right]. \tag{A1}$$

Exponentiating the denominators produces

$$I(\hat{\mathbf{y}},\mathbf{v},\mathbf{v}') = -\frac{1}{|\mathbf{y}|}\int_0^\infty dt\int_0^\infty dt'\int \frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2}e^{i\mathbf{k}\cdot(\mathbf{v}t+\mathbf{v}'t')}\left[2-e^{i\mathbf{k}\cdot\mathbf{y}}-e^{-i\mathbf{k}\cdot\mathbf{y}}\right]. \tag{A2}$$

Completing the $\mathbf{k}$ integral gives

$$I(\mathbf{y},\mathbf{v},\mathbf{v}') = -\frac{1}{|\mathbf{y}|}\int_0^\infty dt\int_0^\infty dt'\left[\frac{2}{|\mathbf{v}t+\mathbf{v}'t'|}-\frac{1}{|\mathbf{v}t+\mathbf{v}'t'+\mathbf{y}|}-\frac{1}{|\mathbf{v}t+\mathbf{v}'t'-\mathbf{y}|}\right]. \tag{A3}$$

We can insert another integral over s and change the variables $t=s\alpha, t'=s\beta$ to get

$$\begin{aligned}
I(\hat{\mathbf{y}},\mathbf{v},\mathbf{v}') &= -\int_0^\infty \frac{ds}{|\mathbf{y}|}\delta(s{-}t{-}t')\int_0^\infty dt\int_0^\infty dt'\left[\frac{2}{|\mathbf{v}t{+}\mathbf{v}'t'|}-\frac{1}{|\mathbf{v}t{+}\mathbf{v}'t'{+}\mathbf{y}|}-\frac{1}{|\mathbf{v}t{+}\mathbf{v}'t'{-}\mathbf{y}|}\right]\\
&= -\int_0^\infty \frac{ds}{|\mathbf{y}|}\int_0^1 d\alpha\left[\frac{2}{|\alpha\mathbf{v}{+}(1{-}\alpha)\mathbf{v}'|}-\frac{1}{|\alpha\mathbf{v}{+}(1{-}\alpha)\mathbf{v}'{+}\mathbf{y}/s|}-\frac{1}{|\alpha\mathbf{v}{+}(1{-}\alpha)\mathbf{v}'{-}\mathbf{y}/s|}\right]\\
&= \int_0^1 d\alpha\frac{1}{[\alpha\mathbf{v}{+}(1{-}\alpha)\mathbf{v}']^2}\int_0^\infty ds\left[\frac{s}{\sqrt{1{-}2s\cos\theta(\alpha){+}s^2}}+\frac{s}{\sqrt{1{+}2s\cos\theta(\alpha){+}s^2}}-2\right]\\
&= \int_0^1 d\alpha\frac{1}{[\alpha\mathbf{v}{+}(1{-}\alpha)\mathbf{v}']^2}\left[2+\cos\theta(\alpha)\ln\frac{1+\cos\theta(\alpha)}{1-\cos\theta(\alpha)}\right],
\end{aligned} \tag{A4}$$

where $\theta(\alpha)$ is the angle between vectors $\alpha\mathbf{v}{+}(1{-}\alpha)\mathbf{v}'$ and $\mathbf{y}$.

We now evaluate integral

$$I(\eta) \equiv \int_0^\infty \frac{dx}{x}\frac{2}{e^x-1}(1-\cos\eta x). \tag{A5}$$

Expanding the denominator gives

$$I(\eta) = \sum_{n=1}^\infty\int_0^\infty \frac{dx}{x}\left[2e^{-nx}-\left(e^{-nx+i\eta x}-e^{-nx-i\eta x}\right)\right] = \sum_{n=1}^\infty \ln\left(1+\frac{\eta^2}{n^2}\right) = \ln\frac{\sinh\pi\eta}{\pi\eta}, \tag{A6}$$

where we have used

$$\int_0^\infty \frac{dx}{x}(e^{-ax}-e^{-bx}) = \ln\frac{a}{b} \tag{A7}$$

and the infinite product

$$\sinh x = x\prod_{n=1}^\infty\left(1{+}\frac{x^2}{\pi^2 n^2}\right). \tag{A8}$$

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